

The bicomplex quantum Coulomb potential problem

J  r  mie Mathieu,¹ Louis Marchildon¹ and Dominic Rochon²

¹*D  partement de physique, Universit   du Qu  bec,
Trois-Rivi  res, Qc, Canada, G9A 5H7*
jeremie.mathieu@uqtr.ca, louis.marchildon@uqtr.ca

²*D  partement de math  matiques et d'informatique,
Universit   du Qu  bec, Trois-Rivi  res, Qc, Canada, G9A 5H7*
dominic.rochon@uqtr.ca

Abstract

Generalizations of the complex number system underlying the mathematical formulation of quantum mechanics have been known for some time, but the use of the commutative ring of bicomplex numbers for that purpose is relatively new. This paper provides an analytical solution of the quantum Coulomb potential problem formulated in terms of bicomplex numbers. We define the problem by introducing a bicomplex hamiltonian operator and extending the canonical commutation relations to a form like $[X_i, P_k] = \mathbf{i}_1 \hbar \xi \delta_{ik} I$, where ξ is a bicomplex number. Following Pauli's algebraic method, we find the eigenvalues of the bicomplex hamiltonian. These eigenvalues are also obtained, along with appropriate eigenfunctions, by solving the extension of Schr  dinger's time-independent differential equation. Examples of solutions are displayed. There is an orthonormal system of solutions that belongs to a bicomplex Hilbert space.

1 Introduction

It is generally believed that the best justification of a physical theory, no matter what the nature of its mathematical formalism, rests on the agreement of its predictions with experiments and the internal consistency or elegance of the theory itself [1]. Views on the mathematical formalism range from pure instrumentalism to the idea that the formalism itself is real [2].

The mathematical formalism of quantum mechanics has been studied thoroughly [3, 4, 5]. Not everyone agrees on the set of postulates necessary to build the foundations of a coherent quantum mechanics [6, 7]. The Hilbert space structure of the set of quantum states, however, seems to be uncontroversial.

Hilbert spaces used in quantum mechanics are defined over the nonordered field of complex numbers \mathbb{C} . Complex numbers make up a division algebra richer than their real subset, and are deeply connected with superposition of quantum-mechanical amplitudes. Unitary representations of Lie groups, fundamental tools in the quantum theory of symmetry, require complex numbers in an essential way [8, 9].

Quantum mechanics postulates that the only possible results of the measurement of a dynamical variable are the eigenvalues of the corresponding self-adjoint operator acting in the state space. As eigenvalues of self-adjoint operators are real, this is a form of correspondence between the quantum and classical world descriptions.

If complex numbers are so appropriate to describe the quantum world, one can ask whether generalizations of that number system might do equally well or even better. The noncommutative field of quaternions has already been investigated from that point of view [10]. More recently, attention has turned towards the commutative ring of bicomplex numbers [11, 12]. This paper is part of a program of extending the quantum mathematical formalism to that algebraic structure, which is neither a division algebra nor an absolute-valued algebra over the real numbers [13, 14, 15, 16, 17, 18].

In sect. 2, we review some of the algebraic properties of bicomplex numbers and construct an infinite-dimensional bicomplex Hilbert space made up of square-integrable bicomplex functions. Section 3 defines the bicomplex generalization of the quantum-mechanical Coulomb potential problem. Eigenvalues of the corresponding hamiltonian are obtained through Pauli's algebraic method. Section 4 is devoted to obtaining eigenfunctions of the bicomplex Coulomb potential hamiltonian in the coordinates basis. To our knowledge, this is the first time that this has been done with an algebra larger than \mathbb{C} . Graphical representations of some functions of interest are shown in sect. 5. In sect. 6 we check the consistency of some of the assumptions made and show that the Coulomb eigenfunctions live in a bicomplex Hilbert space. We conclude in sect. 7.

2 Bicomplex numbers and functions

In this section we briefly summarize relevant algebraic properties of bicomplex numbers. More information and proofs can be found in [11, 12]. We define the concept of a square-integrable bicomplex function. Such functions are then used to construct an infinite-dimensional bicomplex Hilbert space which, it turns out, will be an appropriate arena for bicomplex Coulomb potential eigenfunctions.

2.1 Algebraic structure

One way to define a bicomplex number α is by writing

$$\alpha := \alpha_{\hat{1}}\mathbf{e}_1 + \alpha_{\hat{2}}\mathbf{e}_2, \quad (1)$$

where $\alpha_{\hat{1}}$ and $\alpha_{\hat{2}}$ belong to $\mathbb{C}(\mathbf{i}_1)$, the field of complex numbers. The caret notation [15] is used to label complex components of bicomplex numbers, thereby avoiding confusion with other kinds of indices. The imaginary bicomplex units \mathbf{e}_1 and \mathbf{e}_2 satisfy the remarkable properties

$$\mathbf{e}_1^2 = \mathbf{e}_1, \quad \mathbf{e}_2^2 = \mathbf{e}_2, \quad \mathbf{e}_1 + \mathbf{e}_2 = 1 \quad \text{and} \quad \mathbf{e}_1\mathbf{e}_2 = 0 = \mathbf{e}_2\mathbf{e}_1. \quad (2)$$

We call $\{\mathbf{e}_1, \mathbf{e}_2\}$ the idempotent basis of bicomplex numbers. With the addition and multiplication defined in the obvious way, the set of bicomplex numbers \mathbb{T} forms a commutative ring with unity. Properties (2) greatly simplify bicomplex algebraic calculations and make definition (1), among other equivalent choices, a very useful one for our purposes.

If $\alpha_{\hat{1}} = 0$, then (2) implies that $\alpha\mathbf{e}_1 = 0$. In fact any bicomplex number α for which either $\alpha_{\hat{1}} = 0$ or $\alpha_{\hat{2}} = 0$ is a zero divisor. The set of all zero divisors is called the null cone and is denoted by \mathcal{NC} .

Define \mathbf{j} , the imaginary hyperbolic unit, as $\mathbf{j} := \mathbf{e}_1 - \mathbf{e}_2$. Then $\mathbf{j}^2 = 1$, $\mathbf{e}_1 = (1 + \mathbf{j})/2$ and $\mathbf{e}_2 = (1 - \mathbf{j})/2$. Substituting this in (1), we get the hyperbolic representation of α as

$$\alpha = \left\{ \frac{\alpha_{\hat{1}} + \alpha_{\hat{2}}}{2} \right\} + \left\{ \frac{\alpha_{\hat{1}} - \alpha_{\hat{2}}}{2} \right\} \mathbf{j} =: x_{\alpha} + y_{\alpha} \mathbf{j}, \quad (3)$$

which in turn means that $\alpha_{\hat{1}} = x_{\alpha} + y_{\alpha}$ and $\alpha_{\hat{2}} = x_{\alpha} - y_{\alpha}$. If $\alpha_{\hat{1}}$ and $\alpha_{\hat{2}}$ are both in \mathbb{R} , we call α a hyperbolic number. The set \mathbb{D} of all hyperbolic

numbers is obviously a subset of \mathbb{T} .¹

There are several ways to define conjugation in \mathbb{T} , but the one most useful for our purposes is the following. We define α^\dagger as $\overline{\alpha_1} \mathbf{e}_1 + \overline{\alpha_2} \mathbf{e}_2$, where the upper bar denotes the usual complex conjugation. Clearly,

$$\alpha^\dagger \alpha = |\alpha_1|^2 \mathbf{e}_1 + |\alpha_2|^2 \mathbf{e}_2, \quad (4)$$

where $|\cdot|$ is the standard real norm of complex algebra.

The real norm of a bicomplex number α is defined as

$$|\alpha| := \frac{1}{\sqrt{2}} \sqrt{|\alpha_1|^2 + |\alpha_2|^2}. \quad (5)$$

If α is hyperbolic, then

$$|\alpha| = \sqrt{x_\alpha^2 + y_\alpha^2} := \sqrt{\operatorname{Re}(\alpha)^2 + \operatorname{Hy}(\alpha)^2}.$$

One can show [11] that for all $\alpha, \beta \in \mathbb{T}$ and $z \in \mathbb{C}(\mathbf{i}_1)$,

$$|\alpha| \geq 0, \quad |z\alpha| = |z| |\alpha|, \quad |\alpha + \beta| \leq |\alpha| + |\beta| \quad \text{and} \quad |\alpha\beta| \leq \sqrt{2} |\alpha| |\beta|.$$

Since \mathbb{T} possesses zero divisors and since the norm of a product of bicomplex numbers is not in general equal to the product of their norms, the algebraic structure $(\mathbb{T}, +, \cdot, |\cdot|)$ is neither a division algebra nor an absolute-valued algebra over the real numbers.

2.2 Normed function space

Defining a bicomplex function f of q bicomplex variables as a q -tuple infinite positive-integer power series with bicomplex coefficients, one can show that

$$f(\boldsymbol{\mu}) = f_1(\boldsymbol{\mu}_1) \mathbf{e}_1 + f_2(\boldsymbol{\mu}_2) \mathbf{e}_2. \quad (6)$$

The notation $f(\boldsymbol{\mu})$ means that f depends on q bicomplex variables μ_i , and each f_s ($s = 1, 2$) is a $\mathbb{C}(\mathbf{i}_1)$ complex functions of the q complex variables $\mu_{i\hat{s}}$.

We say that f in (6) belongs to the null cone if either f_1 or f_2 is zero. We call f a hyperbolic function if f_1 and f_2 are both real.

¹Note that $(-\mathbf{i}_1 \mathbf{j})^2 = -1$, so that $-\mathbf{i}_1 \mathbf{j}$ has the properties of an imaginary unit. It is usually called \mathbf{i}_2 .

As a particular case, if all μ_i are real we simply have

$$f(\boldsymbol{\mu}) = f_1(\boldsymbol{\mu}) \mathbf{e}_1 + f_2(\boldsymbol{\mu}) \mathbf{e}_2. \quad (7)$$

We say that f is a bicomplex square-integrable function if and only if the $f_{\hat{s}}$ are both square-integrable functions, that is,

$$\int |f_{\hat{s}}(\boldsymbol{\mu})|^2 d\boldsymbol{\mu} < \infty \quad (8)$$

for $s = 1$ and 2 . Here $d\boldsymbol{\mu}$ is the Lebesgue measure on \mathbb{R}^q [19]. We denote by \mathcal{F}_q the set of bicomplex square-integrable functions of q real variables. It can be shown that with standard addition and multiplication, \mathcal{F}_q makes up a \mathbb{T} -module. This module is explicitly denoted as $(\mathcal{F}_q, \mathbb{T}, +, \cdot)$ and it obviously has infinite dimension.

For any $f, g \in \mathcal{F}_q$, the following binary mapping takes two bicomplex square-integrable functions and transforms them into a unique bicomplex number:

$$(f, g) := \int f^\dagger(\boldsymbol{\mu}) g(\boldsymbol{\mu}) d\boldsymbol{\mu} = \sum_s \mathbf{e}_s \int \overline{f_{\hat{s}}(\boldsymbol{\mu})} g_{\hat{s}}(\boldsymbol{\mu}) d\boldsymbol{\mu}. \quad (9)$$

If we identify functions that differ only on a set of measure zero, the binary mapping (9) satisfies all the properties of a scalar product. Explicitly,

1. $(f, g + h) = (f, g) + (f, h)$;
2. $(f, \alpha g) = \alpha(f, g)$;
3. $(f, g) = (g, h)^\dagger$;
4. $(f, f) = 0$ if and only if $f = 0$.

The functions f and g are orthogonal if their scalar product vanish. We say that f is normalized if $(f, f) = 1$.

With (9), one can define an induced \mathbb{T} -norm on \mathcal{F}_q as

$$\|f\| := \frac{1}{\sqrt{2}} \sqrt{(f, f)_{\hat{1}} + (f, f)_{\hat{2}}} = \frac{1}{\sqrt{2}} \sqrt{\sum_s \int |f_{\hat{s}}(\boldsymbol{\mu})|^2 d\boldsymbol{\mu}}. \quad (10)$$

Making use of results proved in [17], it is not difficult to show that the structure $(\mathcal{F}_q, \mathbb{T}, +, \cdot, (\ , \), \| \ \|)$ is a bicomplex Hilbert space.

3 The Coulomb potential problem

In standard quantum mechanics, the hamiltonian associated with the Coulomb potential is given by

$$H = \frac{1}{2\mu}P^2 - \frac{Ze^2}{R}, \quad (11)$$

where μ , e^2 and Z are positive real numbers and

$$P^2 := P_1^2 + P_2^2 + P_3^2, \quad R := \sqrt{X_1^2 + X_2^2 + X_3^2}. \quad (12)$$

The six operators X_i and P_k are self-adjoint and satisfy the commutation relation $[X_i, P_k] = \mathbf{i}_1 \hbar \delta_{ik} I$.

The quantum-mechanical Coulomb problem consists in finding the eigenvalues and eigenvectors of H , that is, it consists in solving the equation $H|\psi_E\rangle = E|\psi_E\rangle$. Its most important application is the determination of the energy levels and state vectors of a hydrogen atom or hydrogen-like ion in its center-of-mass frame. The Coulomb problem is one of the few analytically solvable problems of quantum mechanics. The solution can be obtained both by an algebraic method that goes back to Pauli [20, 21, 22], and by a differential equation method that goes back to Schrödinger [23, 24, 25].

3.1 Statement of the bicomplex problem

The quantum Coulomb problem will now be formulated in terms of bicomplex numbers. The crucial step consists in extending the canonical commutation relations the way it was done for the harmonic oscillator [15]. We proceed by making a set of assumptions, from which we will derive a number of properties satisfied by eigenvalues and eigenkets of the hamiltonian. The consistency of the assumptions will eventually be checked through the explicit solutions obtained. So here are our assumptions:

- a) Seven bicomplex linear operators X_i , P_k and H , related by (11) and (12), act in a \mathbb{T} -module \mathcal{M} . Elements of \mathcal{M} are called kets and are generically denoted as $|\psi\rangle$. Operators and kets can be decomposed in the idempotent basis as

$$X_i = X_{i1}\mathbf{e}_1 + X_{i2}\mathbf{e}_2, \quad (13)$$

$$|\psi\rangle = |\psi\rangle_1\mathbf{e}_1 + |\psi\rangle_2\mathbf{e}_2. \quad (14)$$

Here $X_{i\mathbf{s}} := \mathbf{e}_{\mathbf{s}} X_i$ and $|\psi\rangle_{\mathbf{s}} := \mathbf{e}_{\mathbf{s}} |\psi\rangle$. We say that X_i belongs to the null cone if either $X_{i\mathbf{1}}$ or $X_{i\mathbf{2}} = 0$. The same applies to P_k and H . Similarly, $|\psi\rangle \in \mathcal{NC}$ if $|\psi\rangle_{\mathbf{1}}$ or $|\psi\rangle_{\mathbf{2}} = |0\rangle$. The bold index notation is such that any quantity affected by a bold subscript \mathbf{s} is unchanged when multiplied by $\mathbf{e}_{\mathbf{s}}$.

- b) The operators X_i and P_k are self-adjoint with respect to a bicomplex scalar product to be specified explicitly. The scalar product has to satisfy properties analogous to the ones enumerated after eq. (9). Self-adjointness is denoted as $X_i = X_i^*$ and $P_k = P_k^*$.
- c) The scalar product of a ket with itself belongs to $\mathbb{D}_+ := \{\alpha_{\mathbf{1}}\mathbf{e}_{\mathbf{1}} + \alpha_{\mathbf{2}}\mathbf{e}_{\mathbf{2}} : \alpha_{\mathbf{1}}, \alpha_{\mathbf{2}} \geq 0\}$.
- d) $[X_i, P_k] = \mathbf{i}_{\mathbf{1}} \hbar \delta_{ik} \xi I$, where $\xi \in \mathbb{T}$ is not in the null cone, I is the identity operator on \mathcal{M} , \hbar is Planck's reduced constant and δ_{ik} is Kronecker's delta.
- e) There are eigenkets $|\psi_E\rangle$ of H which are not in the null cone and whose corresponding eigenvalues E are not in the null cone.
- f) Eigenkets $|\psi_E\rangle$ corresponding to a given eigenvalue E span a finite-dimensional \mathbb{T} -module.
- g) Two eigenkets $|\psi_{E_i}\rangle, |\psi_{E_j}\rangle$ of H , not in \mathcal{NC} , with $(E_i - E_j)$ not in \mathcal{NC} , are orthogonal.

Assumption (a) introduces the bicomplex generalization of the position, momentum and energy operators. With (b) we impose, as in the standard case, the self-adjointness of X_i and P_k . The third general property of the scalar product, stated after eq. (9), implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. The more restrictive assumption (c) is added so that a ket $|\psi\rangle \notin \mathcal{NC}$ can always be normalized [16], through multiplication by $(|\psi\rangle, |\psi\rangle)^{-1/2}$.

The second and third general properties of the scalar product imply that $(\alpha|\psi\rangle, |\phi\rangle) = \alpha^\dagger (|\psi\rangle, |\phi\rangle)$. This means that the eigenvalues of a self-adjoint bicomplex operator (associated with an eigenket that is not in the null cone) are hyperbolic numbers [14].

The simplest possible form of a bicomplex extension of the canonical commutation relations seems to be imbedded in (d). This assumption entails that none of the operators X_i and P_k belongs to the null cone. Assumption (e)

implies that H is not in \mathcal{NC} . Indeed if H_1 vanished, for instance, H could not have an eigenvalue with $E_{\hat{1}} \neq 0$. Making use of the self-adjointness of X_i and P_k and the properties of the scalar product, one can show that the bicomplex number ξ introduced in (d) is in fact a hyperbolic number [15].

Assumption (f) is not really necessary. In effect it restricts the eigenvalues we will obtain to the discrete spectrum. Assumption (g) can be seen as contributing to the specification of the scalar product. That assumption is not necessary to derive eigenvalues through the algebraic method, but it is needed to give structure to the \mathbb{T} -module of eigenkets. In the differential equation method that will be used in sect. 4, (g) will in fact be derived.

We will now show that, without loss of generality, ξ can be taken in \mathbb{D}^+ .² To do this, we show that a simple rescaling of X_i and P_k transforms the problem specified in assumptions (a) to (g) into an equivalent one, but with ξ in \mathbb{D}^+ .

For $s = 1, 2$, let $\alpha_{\hat{s}}$ and $\beta_{\hat{s}}$ be in $\mathbb{C}(\mathbf{i}_1)$ and nonzero. Define X'_i and P'_k so that

$$X_i := (\alpha_{\hat{1}}\mathbf{e}_1 + \alpha_{\hat{2}}\mathbf{e}_2) X'_i \quad \text{and} \quad P_k := (\beta_{\hat{1}}\mathbf{e}_1 + \beta_{\hat{2}}\mathbf{e}_2) P'_k. \quad (15)$$

We have

$$\begin{aligned} (|\phi\rangle, X_i|\psi\rangle) &= \{\alpha_{\hat{1}}\mathbf{e}_1 + \alpha_{\hat{2}}\mathbf{e}_2\} (|\phi\rangle, X'_i|\psi\rangle), \\ (X_i|\phi\rangle, |\psi\rangle) &= \{\overline{\alpha_{\hat{1}}}\mathbf{e}_1 + \overline{\alpha_{\hat{2}}}\mathbf{e}_2\} (X'_i|\phi\rangle, |\psi\rangle). \end{aligned}$$

Since X_i is self-adjoint, the left-hand sides of these two expressions are equal. The right-hand sides must also be equal. Using that and requiring X'_i to be self-adjoint, we find

$$\{\alpha_{\hat{1}}\mathbf{e}_1 + \alpha_{\hat{2}}\mathbf{e}_2\} (|\phi\rangle, X'_i|\psi\rangle) = \{\overline{\alpha_{\hat{1}}}\mathbf{e}_1 + \overline{\alpha_{\hat{2}}}\mathbf{e}_2\} (|\phi\rangle, X'_i|\psi\rangle).$$

Since X'_i is not in \mathcal{NC} , one can always find kets $|\phi\rangle$ and $|\psi\rangle$ such that $(|\phi\rangle, X'_i|\psi\rangle)$ is not in \mathcal{NC} . Therefore $\alpha_{\hat{s}} = \overline{\alpha_{\hat{s}}}$. By a similar argument, $\beta_{\hat{s}} = \overline{\beta_{\hat{s}}}$.

Let us now substitute (15) into (11). In the idempotent basis we have

$$H = \sum_s \left\{ \frac{1}{2\mu} \sum_k \beta_{\hat{s}}^2 P_{k\hat{s}}'^2 - Ze^2 \left[\sum_i \alpha_{\hat{s}}^2 X_{i\hat{s}}'^2 \right]^{-1/2} \right\} \mathbf{e}_s. \quad (16)$$

² \mathbb{D}^+ differs from \mathbb{D}_+ introduced in assumption (c) in that in \mathbb{D}^+ , vanishing values of $\alpha_{\hat{1}}$ and $\alpha_{\hat{2}}$ are excluded.

Sums over indices like i, j and k run from 1 to 3, whereas the idempotent-basis index s runs from 1 to 2. We want H to have a form similar to (11) or, again in the idempotent basis,

$$H = \sum_s \left\{ \frac{1}{2\mu'} \sum_k P_{ks}'^2 - (Ze^2)' \left[\sum_i X_{is}'^2 \right]^{-1/2} \right\} \mathbf{e}_s. \quad (17)$$

Comparing (16) and (17), we see that $\mu' = \mu/\beta_s^2$. This holds for both $s = 1$ and 2. Hence $\beta_1^2 = \beta_2^2$ or, equivalently, $\beta_1 = \pm\beta_2$. Comparing again (16) and (17), we have

$$(Ze^2)' \left[\sum_i X_{is}'^2 \right]^{-1/2} = Ze^2 \left[\sum_i \alpha_s^2 X_{is}'^2 \right]^{-1/2}.$$

Once more, the only way $(Ze^2)'$ can be a real positive number is if $\alpha_1^2 = \alpha_2^2$, or $\alpha_1 = \pm\alpha_2$.

Now assumption (d) and definition (15) allow us to write

$$[X_i, P_k] = \mathbf{i}_1 \hbar \delta_{ik} (\xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2) I = (\alpha_1 \beta_1 \mathbf{e}_1 + \alpha_2 \beta_2 \mathbf{e}_2) [X_i', P_k'].$$

This implies that

$$[X_i', P_k'] = \mathbf{i}_1 \hbar \delta_{ik} \left(\frac{\xi_1}{\alpha_1 \beta_1} \mathbf{e}_1 + \frac{\xi_2}{\alpha_2 \beta_2} \mathbf{e}_2 \right) I =: \mathbf{i}_1 \hbar \delta_{ik} (\xi_1' \mathbf{e}_1 + \xi_2' \mathbf{e}_2) I.$$

Therefore, we can always choose α_s and β_s so that ξ_1' and ξ_2' are real and positive. Moreover, we can rescale ξ_1' to be 1, but we cannot in general rescale ξ_2' so that $\xi_2' = \xi_1'$. We conclude that H can always be written as in (11), with the commutation relations between X_i and P_k given by

$$[X_i, P_k] = \mathbf{i}_1 \hbar \delta_{ik} (\xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2) I, \quad \xi_1, \xi_2 \in \mathbb{R}^+. \quad (18)$$

3.2 Eigenvalues of H

In this section, eigenvalues of H are obtained through Pauli's algebraic method. This shows the unicity of the eigenvalues obtained, under assumptions (a)–(g).

Just as in standard quantum mechanics, we define the bicomplex angular momentum operator $\mathbf{L} := \mathbf{R} \times \mathbf{P}$. In terms of the Levi-Civita symbol we have

$$L_i = \sum_{jk} \epsilon_{ijk} X_j P_k = \sum_s \left\{ \sum_{jk} \epsilon_{ijk} X_{js} P_{ks} \right\} \mathbf{e}_s. \quad (19)$$

The bicomplex Runge-Lenz vector is defined as

$$\mathbf{A} := \frac{1}{2\mu} (\mathbf{P} \times \mathbf{L} - \mathbf{L} \times \mathbf{P}) - Ze^2 \frac{\mathbf{R}}{R},$$

which is equivalent to

$$A_i = \sum_s \left\{ \frac{1}{\mu} \left(\sum_{jk} \epsilon_{ijk} P_{js} L_{ks} - \mathbf{i}_1 \hbar \xi_{\hat{s}} P_{is} \right) - Ze^2 \frac{X_{is}}{R_s} \right\} \mathbf{e}_s. \quad (20)$$

Let us write $\eta := \hbar \xi$. We then observe that the commutator of X_{is} and P_{js} , as well as the definitions of L_{is} and A_{js} , are the same as the ones in the standard quantum-mechanical case, except that \hbar is everywhere replaced by $\eta_{\hat{s}}$. By an argument identical as in the standard case [21, 22], we therefore obtain the following commutation relations and properties:

$$[A_{is}, H_s] = 0, \quad (21)$$

$$[L_{is}, A_{js}] = \mathbf{i}_1 \eta_{\hat{s}} \sum_k \epsilon_{ijk} A_{ks}, \quad (22)$$

$$[A_{is}, A_{js}] = -\frac{2\mathbf{i}_1 \eta_{\hat{s}}}{\mu} H_s \sum_k \epsilon_{ijk} L_{ks}, \quad (23)$$

$$\sum_i L_{is} A_{is} = 0 = \sum_i A_{is} L_{is}, \quad (24)$$

$$A_s^2 = (Ze^2)^2 + \frac{2}{\mu} H_s \{L_s^2 + \eta_{\hat{s}}^2\}. \quad (25)$$

Here $L_s^2 = \sum_i L_{is}^2$ and $A_s^2 = \sum_i A_{is}^2$. By properties of the idempotent basis, operators H , L_i , and A_j satisfy similar relations as (21)–(25), with the index \mathbf{s} deleted. Note that, because of (23), operators H_s , L_{is} , and A_{js} do not make up a Lie algebra. They do, however, generate an infinite-dimensional one.

To avoid having to work with (the bicomplex generalization of) an infinite-dimensional algebra, we will restrict the action of H , L_i , and A_j on the module \mathcal{F}_E corresponding to a given eigenvalue E of H . By assumption (f), \mathcal{F}_E is finite-dimensional. Assuming as in (e) that E is not in \mathcal{NC} , we define three operators \tilde{A}_i acting on \mathcal{F}_E as

$$\tilde{A}_i := \sqrt{\frac{-\mu}{2E}} A_i. \quad (26)$$

Equations (21)–(23) (without the index \mathbf{s}) imply quite straightforwardly that

$$[\tilde{A}_i, H] = 0, \quad (27)$$

$$[L_i, \tilde{A}_j] = \mathbf{i}_1 \eta \sum_k \epsilon_{ijk} \tilde{A}_k, \quad (28)$$

$$[\tilde{A}_i, \tilde{A}_j] = \mathbf{i}_1 \eta \sum_k \epsilon_{ijk} L_k. \quad (29)$$

Making use of (24) and (25), we easily obtain

$$\sum_i L_i \tilde{A}_i = 0 = \sum_i \tilde{A}_i L_i \quad (30)$$

and

$$\sum_i \tilde{A}_i \tilde{A}_i + \sum_j L_j L_j + \eta^2 + \frac{\mu}{2E} (Ze^2)^2 = 0. \quad (31)$$

To arrive at explicit values of E , it is useful to construct six operators F_i and G_i as

$$F_i := \frac{1}{2} (L_i - \tilde{A}_i) \quad \text{and} \quad G_i := \frac{1}{2} (L_i + \tilde{A}_i). \quad (32)$$

Since L_i and \tilde{A}_i commute with H , so do F_i and G_i . The commutation relations of the latter are given by

$$[F_i, F_j] = \mathbf{i}_1 \eta \sum_k \epsilon_{ijk} F_k, \quad (33)$$

$$[G_i, G_j] = \mathbf{i}_1 \eta \sum_k \epsilon_{ijk} G_k, \quad (34)$$

$$[F_i, G_j] = 0. \quad (35)$$

This means that the F_i commute with the G_j , but the F_i (and the G_i) have with themselves the same commutation relation as the bicomplex angular momentum operators. Since F_i and G_j commute, F^2 commutes with G^2 . With (30) and (32), one can show that $F^2 - G^2 = 0$ and, therefore, that their eigenvalues are equal.

If we project (33) on the idempotent basis, we find that for each \mathbf{s} the $F_{i\mathbf{s}}$ have the commutation relations of standard angular momentum (with \hbar replaced by $\eta_{\hat{\mathbf{s}}}$). The eigenvalue of $F_{\mathbf{s}}^2$ is therefore equal to $f_{\hat{\mathbf{s}}}(f_{\hat{\mathbf{s}}} + 1) \eta_{\hat{\mathbf{s}}}^2$, where $f_{\hat{\mathbf{s}}}$ is a nonnegative integer or half integer [26]. The eigenvalues of F^2 (and of G^2) are therefore equal to

$$\sum_{\mathbf{s}} f_{\hat{\mathbf{s}}}(f_{\hat{\mathbf{s}}} + 1) \eta_{\hat{\mathbf{s}}}^2 \mathbf{e}_{\mathbf{s}} = f(f + 1) \eta^2, \quad (36)$$

where of course $f = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2$.

If we substitute (32) in (31) we get

$$0 = 2 \sum_i F_i F_i + 2 \sum_j G_j G_j + \eta^2 + \frac{\mu}{2E} (Ze^2)^2.$$

For the eigenvalues this entails that

$$0 = (2f + 1)^2 \eta^2 + \frac{\mu}{2E} (Ze^2)^2$$

or, in the idempotent basis,

$$0 = \sum_{\mathbf{s}} \left\{ (2f_{\hat{\mathbf{s}}} + 1)^2 \eta_{\hat{\mathbf{s}}}^2 + \frac{\mu}{2E_{\hat{\mathbf{s}}}} (Ze^2)^2 \right\} \mathbf{e}_{\mathbf{s}}.$$

But then

$$(2f_{\hat{\mathbf{s}}} + 1)^2 \eta_{\hat{\mathbf{s}}}^2 + \frac{\mu}{2E_{\hat{\mathbf{s}}}} (Ze^2)^2 = 0$$

for both $s = 1$ and 2 . This yields for $E_{\hat{\mathbf{s}}}$

$$E_{\hat{\mathbf{s}}} = -\frac{\mu (Ze^2)^2}{2\eta_{\hat{\mathbf{s}}}^2 (2f_{\hat{\mathbf{s}}} + 1)^2} = -\frac{\mu (Ze^2)^2}{2\hbar^2 \xi_{\hat{\mathbf{s}}}^2 (2f_{\hat{\mathbf{s}}} + 1)^2} =: -\frac{\mu Z^2 e^4}{2\hbar^2 \xi_{\hat{\mathbf{s}}}^2 n_{\hat{\mathbf{s}}}^2},$$

where $n_{\hat{\mathbf{s}}}$ is a positive integer. This means that we can write

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 \xi^2 n^2} = \sum_{\mathbf{s}} \left\{ -\frac{\mu Z^2 e^4}{2\hbar^2 \xi_{\hat{\mathbf{s}}}^2 n_{\hat{\mathbf{s}}}^2} \right\} \mathbf{e}_{\mathbf{s}}. \quad (37)$$

This coincides with the standard Coulomb potential energy levels [24, 25] if and only if $\xi_{\hat{1}} = 1 = \xi_{\hat{2}}$ and $n_{\hat{1}} = n_{\hat{2}} = n$. We call $n = n_{\hat{1}}\mathbf{e}_1 + n_{\hat{2}}\mathbf{e}_2$ the bicomplex principal quantum number. In the hyperbolic representation,

$$E_n = -\frac{\mu Z^2 e^4}{4\hbar^2} \left\{ \left[\xi_{\hat{1}}^{-2} n_{\hat{1}}^{-2} + \xi_{\hat{2}}^{-2} n_{\hat{2}}^{-2} \right] + \left[\xi_{\hat{1}}^{-2} n_{\hat{1}}^{-2} - \xi_{\hat{2}}^{-2} n_{\hat{2}}^{-2} \right] \mathbf{j} \right\}. \quad (38)$$

We point out a formal symmetry satisfied by (38). Since $\mathbf{j} := \mathbf{e}_1 - \mathbf{e}_2$, we can see that $\sqrt{\mathbf{j}} = \mathbf{e}_1 + \mathbf{i}_1 \mathbf{e}_2$. But then $\xi \sqrt{\mathbf{j}} = \xi_{\hat{1}} \mathbf{e}_1 + \mathbf{i}_1 \xi_{\hat{2}} \mathbf{e}_2$. This immediately implies that

$$\mathbb{R}\mathrm{e}(E_n, \xi) = \mathbb{H}\mathrm{y}\left(E_n, \xi \sqrt{\mathbf{j}}\right) \quad \text{and} \quad \mathbb{R}\mathrm{e}\left(E_n, \xi \sqrt{\mathbf{j}}\right) = \mathbb{H}\mathrm{y}(E_n, \xi).$$

Note, however, that $\xi \sqrt{\mathbf{j}}$ is not a hyperbolic number.

4 Eigenfunctions of H

In this section, we define a coordinate-basis representation for the bicomplex operators X_i and P_k . We then show that they, as well as H , are self-adjoint with respect to the scalar product defined in sect. 2.2. Finally, solving the hamiltonian eigenvalue equation in the coordinate basis, we recover eigenvalues given in (37) and obtain the hyperbolic Coulomb potential eigenfunctions.

4.1 Coordinate-basis representation

We begin by constructing a representation of X_i and P_k on \mathcal{F}_3 , the space of bicomplex square-integrable functions on \mathbb{R}^3 . Letting \mathbf{r} denote the triplet (x_1, x_2, x_3) , we define the action of X_i as

$$X_i f(\mathbf{r}) := x_i f(\mathbf{r}). \quad (39)$$

This implies that the X_i commute two by two. A function $F(\mathbf{R})$ acts on f as

$$F(\mathbf{R}) f(\mathbf{r}) := F(\mathbf{r}) f(\mathbf{r}). \quad (40)$$

For the action of P_k we write, in a rather straightforward extension of the standard case

$$P_k f(\mathbf{r}) := -\mathbf{i}_1 \hbar \xi \frac{\partial}{\partial x_k} f(\mathbf{r}) = -\mathbf{i}_1 \eta \frac{\partial}{\partial x_k} f(\mathbf{r}), \quad (41)$$

where $\xi = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2$ and the $\xi_{\hat{s}}$ are positive real numbers.³ Clearly the P_k commute two by two. Moreover, by letting both sides act on an arbitrary function f , one easily shows that

$$[X_i, P_k] = \mathbf{i}_1 \eta \delta_{ik} I. \quad (42)$$

Let f and g be in \mathcal{F}_3 . Clearly $(X_i f, g) = (f, X_i g)$, so that X_i is self-adjoint. For P_k we have

$$\begin{aligned} (P_k f, g) - (f, P_k g) &= \mathbf{i}_1 \eta^\dagger \left(\frac{\partial f}{\partial x_k}, g \right) + \mathbf{i}_1 \eta \left(f, \frac{\partial g}{\partial x_k} \right) \\ &= \mathbf{i}_1 \eta \int \left[\frac{\partial f}{\partial x_k} \right]^\dagger g(\mathbf{r}) \, d\mathbf{r} + \mathbf{i}_1 \eta \int f^\dagger(\mathbf{r}) \frac{\partial g}{\partial x_k} d\mathbf{r} \\ &= \mathbf{i}_1 \eta \int \frac{\partial}{\partial x_k} [f^\dagger(\mathbf{r}) g(\mathbf{r})] \, d\mathbf{r} = 0. \end{aligned}$$

To obtain the last equality, we have restricted the space of functions to those that vanish at infinity and on these functions, P_k is self-adjoint. The proof that $(Hf, g) = (f, Hg)$, and therefore that H is self-adjoint, is straightforward.

4.2 Wave functions

The bicomplex quantum Coulomb potential problem consists in solving the three-dimensional eigenvalue equation

$$H\psi_E(\mathbf{r}) = E\psi_E(\mathbf{r}) \quad (43)$$

for H given by (11). Making use of (40) and (41) we can write more explicitly

$$-\left\{ \frac{\eta^2}{2\mu} \nabla^2 + \frac{Ze^2}{r} \right\} \psi_E(\mathbf{r}) = E\psi_E(\mathbf{r}). \quad (44)$$

We now write ξ , E , and ψ_E in the idempotent basis. Equation (44) becomes

$$\sum_s \left\{ \frac{\eta_s^2}{2\mu} \nabla^2 + \frac{Ze^2}{r} + E_{\hat{s}} \right\} (\psi_E)_{\hat{s}}(\mathbf{r}) \mathbf{e}_s = 0. \quad (45)$$

³Strictly speaking, P_k should be defined on a subset of \mathcal{F}_3 , made up of suitably differentiable functions. We'll come back to this in sect. 6.

Clearly, each coefficient of \mathbf{e}_s must separately vanish. Writing the laplacian in spherical coordinates and making use of the standard expression of the angular momentum operator we get

$$\left\{ \frac{\eta_s^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{L_s^2}{\eta_s^2} \right] + \frac{Ze^2}{r} + E_{\hat{s}} \right\} (\psi_E)_{\hat{s}} = 0, \quad (46)$$

for $s = 1$ and 2 . Now we know from standard quantum mechanics that the spherical harmonics are eigenfunctions of the square of the angular momentum, that is,

$$L_s^2 Y_{l_s m_s} = l_s(l_s + 1) \eta_s^2 Y_{l_s m_s}, \quad (47)$$

with $-l_s \leq m_s \leq l_s$. Looking for solutions of the form

$$(\psi_E)_{\hat{s}}(r, \theta, \phi) := u_{\hat{s}}(r) Y_{l_s m_s}(\theta, \phi) \quad (48)$$

and using (47), we get for $s = 1, 2$

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d}{dr} u_{\hat{s}}(r) \right] - \left[\frac{l_s(l_s + 1)}{r^2} - \frac{2\mu}{\eta_s^2} \left(\frac{Ze^2}{r} + E_{\hat{s}} \right) \right] u_{\hat{s}}(r) = 0. \quad (49)$$

Just as in the standard case, the suitably normalized solutions of (49) are given by [24, 25]

$$u_{n_s l_s}(r) = \left[\left(\frac{2Z}{n_s a_{s0}} \right)^3 \frac{(n_s - l_s - 1)!}{2n_s [(n_s + l_s)!]} \right]^{1/2} e^{-\zeta_s/2} \zeta_s^{l_s} L_{n_s - l_s - 1}^{2l_s + 1}(\zeta_s), \quad (50)$$

where $l_s < n_s$, the $L_{n_s - l_s - 1}^{2l_s + 1}$ are Laguerre polynomials and

$$\zeta_s := \frac{2Z}{n_s a_{s0}} r, \quad a_{s0} := a_0 \xi_s^2 = \frac{\eta_s^2}{\mu e^2}. \quad (51)$$

Each solution corresponds to a value of $E_{\hat{s}}$ given by $-\mu Z^2 e^4 / 2\eta_s^2 n_s^2$. Thus we recover the eigenvalues (37). The bicomplex wave functions in (43) can be written as

$$\psi_{nlm}(\mathbf{r}) = u_{nl}(r) Y_{lm}(\theta, \phi) = \sum_s u_{n_s l_s}(r) Y_{l_s m_s}(\theta, \phi) \mathbf{e}_s. \quad (52)$$

For $\xi_{\hat{1}}$ and $\xi_{\hat{2}}$ fixed, any sextuplet $(n_{\hat{1}}, n_{\hat{2}}, l_{\hat{1}}, l_{\hat{2}}, m_{\hat{1}}, m_{\hat{2}})$ defines an eigenfunction of H . All functions with the same $(n_{\hat{1}}, n_{\hat{2}})$ correspond to the same eigenvalue. A general eigenfunction of H can therefore be written as

$$\sum_s \sum_{l_{\hat{s}}=0}^{n_{\hat{s}}-1} \sum_{m_{\hat{s}}=-l_{\hat{s}}}^{l_{\hat{s}}} C_{l_{\hat{s}}m_{\hat{s}}} u_{n_{\hat{s}}l_{\hat{s}}}(r) Y_{l_{\hat{s}}m_{\hat{s}}}(\theta, \phi) \mathbf{e}_s \quad \text{with} \quad C_{l_{\hat{s}}m_{\hat{s}}} \in \mathbb{T}. \quad (53)$$

Along the way, we have introduced the bicomplex orbital quantum number $l = l_{\hat{1}}\mathbf{e}_1 + l_{\hat{2}}\mathbf{e}_2$ and the bicomplex magnetic quantum number $m = m_{\hat{1}}\mathbf{e}_1 + m_{\hat{2}}\mathbf{e}_2$. When $n_{\hat{1}} = n_{\hat{2}}$, the number n is real, and similarly with l and m .

5 Graphical representation

It is interesting to plot some of the eigenfunctions (52) explicitly. We will concentrate on the radial part. (Note that if $l_{\hat{1}} \neq l_{\hat{2}}$, the spherical harmonics will never factor out as real functions of θ and ϕ .) We write

$$u_{nl}(r) := u_{n_{\hat{1}}l_{\hat{1}}}\mathbf{e}_1 + u_{n_{\hat{2}}l_{\hat{2}}}\mathbf{e}_2. \quad (54)$$

It is particularly instructive to use the decomposition of $\{\mathbf{e}_1, \mathbf{e}_2\}$ in terms of \mathbf{j} given in sect. 2.1. We can then rewrite (54) as

$$\begin{aligned} u_{nl}(r) = & \frac{1}{2} \sum_s \sqrt{u_{n_{\hat{s}}l_{\hat{s}}}^0} \xi_{\hat{s}}^{-3} e^{-\zeta_{\hat{s}}/2} \zeta_{\hat{s}}^{l_{\hat{s}}} L_{n_{\hat{s}}-l_{\hat{s}}-1}^{2l_{\hat{s}}+1}(\zeta_{\hat{s}}) \\ & + \frac{1}{2} \sum_s (-1)^{s+1} \sqrt{u_{n_{\hat{s}}l_{\hat{s}}}^0} \xi_{\hat{s}}^{-3} e^{-\zeta_{\hat{s}}/2} \zeta_{\hat{s}}^{l_{\hat{s}}} L_{n_{\hat{s}}-l_{\hat{s}}-1}^{2l_{\hat{s}}+1}(\zeta_{\hat{s}}) \mathbf{j}, \end{aligned} \quad (55)$$

where $\sqrt{u_{n_{\hat{s}}l_{\hat{s}}}^0} \xi_{\hat{s}}^{-3}$ is the normalization constant.

It will be useful to define, as in (3),

$$\zeta_{\hat{1}} = x_{\zeta} + y_{\zeta}, \quad \zeta_{\hat{2}} = x_{\zeta} - y_{\zeta}, \quad (56)$$

and consider the three functions $\text{Re}(u_{nl})$, $\text{Hy}(u_{nl})$ and $|u_{nl}|$ as depending on r through the two variables x_{ζ} and y_{ζ} . With $\xi_{\hat{s}}$ fixed, graphical representations of these functions can be obtained by assigning specific values to $n_{\hat{s}}$ and $l_{\hat{s}}$.

Figure 1 shows the real part, the hyperbolic part and the real norm of (55) in the specific eigenstate corresponding to $n_{\hat{1}} = 8$, $n_{\hat{2}} = 7$, $l_{\hat{1}} = 6$, $l_{\hat{2}} = 5$.

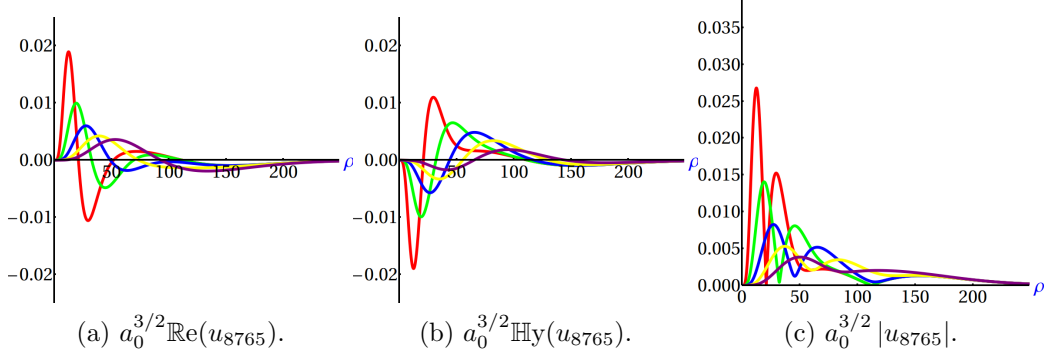


Figure 1: Coulomb potential radial function in a specific eigenstate, for $\xi_2 = 0.50$ (red), 0.62 (green), 0.74 (blue), 0.86 (yellow) and 0.98 (purple).

Here $\xi_1 = 1$ and we have shown the curves obtained for different values of ξ_2 , from 0.50 to 0.98 by intervals of 0.12 . The independent variable ρ is defined as $2Zr/a_0$ and the function plotted is indicated under each graph.

In fig. 2, we go beyond the representation of eigenfunctions of H and extend x_ζ and y_ζ to genuinely independent variables. For ξ fixed, this is equivalent to considering r in the hyperbolic plane. This allows for interesting surfaces to emerge, connected to a new class of polynomials in two real variables. We take $\xi_1 = 1 = \xi_2$ and let $n_{\hat{s}} = 25$ and $l_{\hat{s}} = 12$ for $s = 1, 2$. In these plots, a cut at $y_\zeta = 0$ makes the hyperbolic part of u_{nl} vanish. The real part and real norm then coincide with the ones in the standard case.

There is another equivalent way to generate the radial surfaces in the

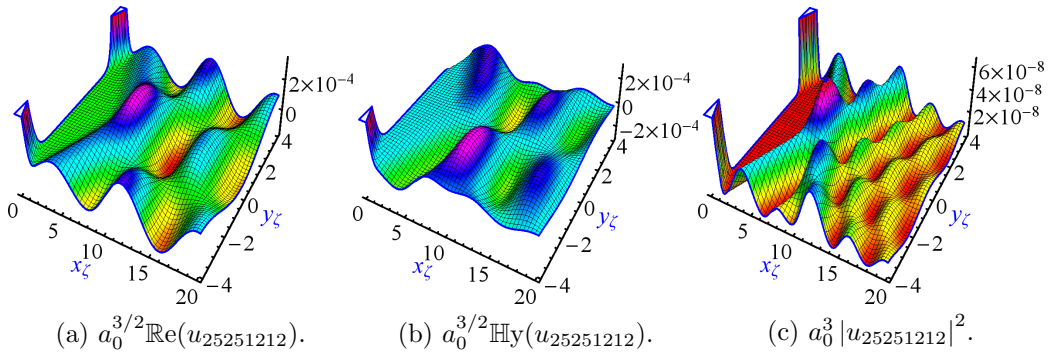


Figure 2: u_{nl} as a function of two independent variables.

hyperbolic basis $\{1, \mathbf{j}\}$ in the standard case. We can write eigenfunctions (54) as

$$u_{nl} = \sqrt{u_{nl}^0} \xi^{-3} e^{-\zeta/2} \ell_{nl}(\zeta),$$

where $\xi := x_\xi + y_\xi \mathbf{j}$, $\zeta := x_\zeta + y_\zeta \mathbf{j}$ and

$$\ell_{nl}(\zeta) := \zeta^l L_{n-l-1}^{2l+1}(\zeta) = \operatorname{Re}(\ell_{nl}(x_\zeta, y_\zeta)) + \operatorname{Hy}(\ell_{nl}(x_\zeta, y_\zeta)) \mathbf{j},$$

where

$$L_{n-l-1}^{2l+1}(\zeta) := \sum_s L_{n_{\hat{s}}-l_{\hat{s}}-1}^{2l_{\hat{s}}+1}(\zeta_{\hat{s}}) \mathbf{e}_s.$$

Using the correspondence between the idempotent and hyperbolic bases, one can easily show that

$$\xi^{-3} = \frac{1}{2} \left\{ (x_\xi + y_\xi)^{-3} + (x_\xi - y_\xi)^{-3} \right\} + \frac{1}{2} \left\{ (x_\xi + y_\xi)^{-3} - (x_\xi - y_\xi)^{-3} \right\} \mathbf{j},$$

which we define as $x'_\xi + y'_\xi \mathbf{j}$. It is not difficult to show that the exponential transforms as

$$e^{-\zeta/2} = e^{-x_\zeta/2} \left\{ \cosh \frac{y_\zeta}{2} - \mathbf{j} \sinh \frac{y_\zeta}{2} \right\}.$$

This suggests that we can explicitly write

$$\begin{aligned} \operatorname{Re}(u_{nl}) &= \sqrt{u_{nl}^0} e^{-x_\zeta/2} \left[\left(x'_\xi \cosh \frac{y_\zeta}{2} - y'_\xi \sinh \frac{y_\zeta}{2} \right) \operatorname{Re}(\ell_{nl}(x_\zeta, y_\zeta)) \right. \\ &\quad \left. + \left(y'_\xi \cosh \frac{y_\zeta}{2} - x'_\xi \sinh \frac{y_\zeta}{2} \right) \operatorname{Hy}(\ell_{nl}(x_\zeta, y_\zeta)) \right], \\ \operatorname{Hy}(u_{nl}) &= \sqrt{u_{nl}^0} e^{-x_\zeta/2} \left[\left(x'_\xi \cosh \frac{y_\zeta}{2} - y'_\xi \sinh \frac{y_\zeta}{2} \right) \operatorname{Hy}(\ell_{nl}(x_\zeta, y_\zeta)) \right. \\ &\quad \left. + \left(y'_\xi \cosh \frac{y_\zeta}{2} - x'_\xi \sinh \frac{y_\zeta}{2} \right) \operatorname{Re}(\ell_{nl}(x_\zeta, y_\zeta)) \right]. \end{aligned}$$

With a symbolic computation software, we can generate the polynomial $\ell_{nl}(\zeta)$ for any positive integers $n_{\hat{s}}$ and $l_{\hat{s}} \leq n_{\hat{s}} - 1$, for $s = 1, 2$. We then take $n = 25$, $l = 12$, $x_\xi = 1$, $y_\xi = 0$ (implying that $x'_\xi = 1$ and $y'_\xi = 0$), compute $\ell_{nl}(\zeta)$, transform ζ into $x_\zeta + y_\zeta \mathbf{j}$ and separate the real and hyperbolic parts to get $\operatorname{Re}(\ell_{nl}(x_\zeta, y_\zeta))$ and $\operatorname{Hy}(\ell_{nl}(x_\zeta, y_\zeta))$ explicitly. If we plot the associated $a_0^{3/2} \operatorname{Re}(u)$, $a_0^{3/2} \operatorname{Hy}(u)$ and $a_0^3 |u|^2$, we recover the results shown in fig. 2.

6 Discussion

We have solved the eigenvalue equation (43) for the discrete spectrum of the Coulomb potential hamiltonian (11) in the framework of bicomplex numbers. The continuous spectrum could also be worked out along similar lines. The eigenvalues corresponding to the discrete spectrum are given in (37) and the eigenfunctions in (52). Note that if $\xi_1 = 1 = \xi_2$, the standard wave functions can be recovered by letting $n_1 = n_2$, $l_1 = l_2$ and $m_1 = m_2$.

It is instructive to investigate the orthogonality properties of the eigenfunctions (52). Making use of definition (9) of the scalar product, one can write

$$\begin{aligned} (\psi_{nlm}, \psi_{n'l'm'}) &= \int \psi_{nlm}^\dagger(\mathbf{r}) \psi_{n'l'm'}(\mathbf{r}) d\mathbf{r} \\ &= \sum_s \mathbf{e}_s \int \psi_{n_s l_s m_s}^\dagger(\mathbf{r}) \psi_{n'_s l'_s m'_s}(\mathbf{r}) d\mathbf{r}. \end{aligned} \quad (57)$$

It is well-known [24, 25] that the standard Coulomb problem eigenfunctions are orthonormal in all indices. This implies that

$$(\psi_{nlm}, \psi_{n'l'm'}) = \sum_s \mathbf{e}_s \delta_{n_s n'_s} \delta_{l_s l'_s} \delta_{m_s m'_s}. \quad (58)$$

From (58) we can draw two conclusions:

1. The eigenfunction ψ_{nlm} is normalized. Indeed

$$(\psi_{nlm}, \psi_{nlm}) = \sum_s \mathbf{e}_s \delta_{n_s n_s} \delta_{l_s l_s} \delta_{m_s m_s} = \sum_s \mathbf{e}_s = 1. \quad (59)$$

2. If $E_n - E_{n'}$ is not in the null cone, then ψ_{nlm} and $\psi_{n'l'm'}$ are orthogonal. Indeed from (37) we see that $E_n - E_{n'}$ is not in the null cone if and only if $n_s \neq n'_s$ for $s = 1, 2$. But then $\delta_{n_s n'_s} = 0$ for $s = 1, 2$ and the orthogonality follows from (58).

Let us consider the set of all finite linear combinations of eigenfunctions ψ_{nlm} , with bicomplex coefficients C_{nlm} . It is straightforward to show that this set makes up a \mathbb{T} -module, which we denote as \mathcal{M} . Defining X_i and P_k as in (39) and (41), one sees that these operators are well-defined on \mathcal{M} . Moreover, it is not difficult to show that properties (a)–(g) in sect. 3.1 are satisfied in \mathcal{M} , thus proving their consistency.

Of course, \mathcal{M} is not a Hilbert space, since the restriction to finite linear combinations entails that it is not complete. It is well-known that the Coulomb potential eigenfunctions in standard quantum mechanics, *i.e.* the $\psi_{n\hat{s}l\hat{s}m\hat{s}}$, make up an orthonormal system in the Hilbert space $L^2(\mathbb{R}^3)$ [5]. From [17], one concludes that the ψ_{nlm} make up an orthonormal system in a bicomplex Hilbert space $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$.

We close this section with a result that we prove with the notation of the Coulomb problem, but that clearly holds more generally. Let U be a bicomplex linear operator, acting on \mathcal{M} , that commutes with H . Then

$$(E_n - E_{n'}) (\psi_{nlm}, U\psi_{n'l'm'}) = 0. \quad (60)$$

The proof is straightforward:

$$\begin{aligned} 0 &= (\psi_{nlm}, [H, U] \psi_{n'l'm'}) \\ &= (\psi_{nlm}, \{HU - UH\} \psi_{n'l'm'}) \\ &= (\psi_{nlm}, HU\psi_{n'l'm'}) - (\psi_{nlm}, UH\psi_{n'l'm'}) \\ &= (U^*H^*\psi_{nlm}, \psi_{n'l'm'}) - (\psi_{nlm}, UH\psi_{n'l'm'}) \\ &= E_n (U^*\psi_{nlm}, \psi_{n'l'm'}) - E_{n'} (\psi_{nlm}, U\psi_{n'l'm'}) \\ &= (E_n - E_{n'}) (\psi_{nlm}, U\psi_{n'l'm'}). \end{aligned}$$

This means that if $E_n - E_{n'}$ is not in the null cone, then $(\psi_{nlm}, U\psi_{n'l'm'})$ vanishes. In other words, $U\psi_{n'l'm'}$ is a linear combination of functions associated with eigenvalue $E_{n'}$.

7 Conclusion

We have shown that, just like the quantum harmonic oscillator problem [15], the quantum Coulomb problem can be solved in the framework of bicomplex numbers. We have obtained the eigenvalues and eigenfunctions of the bicomplex hamiltonian, and have shown that the eigenfunctions make up an orthonormal system in a bicomplex Hilbert space. The question is still open whether the constants ξ_1 and ξ_2 can be given a physical interpretation. In any case, it is likely that the mathematical properties of the functions we introduced can be fruitfully studied for their own sake.

Acknowledgments

LM and DR are grateful to the Natural Sciences and Engineering Research Council of Canada for financial support.

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